Tomographic entropy and cosmology

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Abstract

The probability representation of quantum mechanics including propagators and tomograms of quantum states of the universe and its application to quantum gravity and cosmology are reviewed. The minisuperspaces modeled by oscillator, free pointlike particle and repulsive oscillator are considered. The notion of tomographic entropy and its properties are used to find some inequalities for the tomographic probability determining the quantum state of the universe. The sense of the inequality as a lower bound for the entropy is clarified.

1 Introduction

Recently [1] [2][3][4] a tomographic probability approach to describe the states of the universe in quantum cosmology was suggested. In the framework of this approach the quantum state of the universe is associated with the standard positive probability distribution (function or functional). The probability distribution contains the same information on the universe quantum state that the wave function of the universe [5] [6][7] or the density matrix of the universe [8], [9]. The latter can be presented in different forms, e.g. in form of a Wigner function [10] considered in [11] in a cosmological context. In fact the tomographic probability distribution describing the state of the universe is a symbol of a density operator [12][13] and the tomographic symbols of the operators realize one of the variants of the star-product quantization scheme widely used [14] to study the relation of classical and quantum pictures [15], which can also be applied to study the relation of classical and quantum descriptions of the universe in quantum cosmology. One of the important ingredients of such descriptions is the evolution of the state. In quantum mechanics such evolution is completely described by means of a complex transition probability amplitude from an initial state to a final one. This probability amplitude (propagator) can be presented in the form of a Feynman path integral containing the classical action. In quantum mechanics in the probability representation using the tomographic approach the state evolution can be associated with the standard transition probability. It contains also information on the transition probability amplitude related to the probability by integral transform induced by the Radon transform relating the density matrix (Wigner function) with the quantum tomographic probability [16], [17], [18], [19].

In our previous work [1] we suggested to associate the state of the universe in quantum cosmology with the tomographic probability (or tomogram). The aim of our paper is to consider now in the framework of the suggested probability representation of the universe state in quantum cosmology also the cosmological dynamics and to express this dynamics in terms of a positive transition probability connecting initial and final tomograms of the universe. Another goal of the work is to discuss the tomographic entropy of the quantum state of the universe and a possible experimental approach to observe the tomogram of the universe at its present stage and try to extract some information on the tomogram of the initial state of the universe as well as to find some unilateral constraints (inequalities). The idea of this attempt is based on the fact that tomograms may describe the states of a classical system and the states of its quantum counterpart. In this sense in the probability representation of the quantum state there is not such a dramatic difference between the classical and quantum pictures as the difference between wave function (or density matrix) and classical probability distribution (or trajectory) in the classical phase space. Due to this one can try to study the cosmological dynamics namely in the tomographic probability representation.

In order to illustrate the idea we will use the same simple example of the universe description by means of the minisuperspace discussed, e.g., in [7], [21] . In these minisuperspaces the quantum cosmological dynamics in operative form is reduced to the dynamics of formal quantum systems described by Hamiltonians of the types of oscillator, free motion and free falling particles. In view of this one can apply the same recently obtained results on description of such systems by tomographic probabilities [22] to the cosmological dynamics.

The paper is organized as follows. In the next section we will review the cosmology in terms of a homogeneous (and isotropic) metric with a time dependent parameter the expansion factor of the universe. In section 3, we review the tomographic approach to evolution of the quantum system. In section 4 we consider the examples of the minisuperspace described by the reduced Hamiltonians. In section 5 we study the tomographic entropy and its evolution. Conclusions and perspectives are presented in section 6.

2 The cosmological equations for an homogeneous and isotropic universe

Let us recall briefly the equations for a classical homogeneous and isotropic universe. It is described by one of the following metrics

$$ds^2 = -c^2dt^2 + \frac{a^2}{1-kr^2}\left(dr^2 + r^2d\theta^2 + r^2\sin^2\theta\right)d\phi^2 \tag{2.1}$$

where the parameter k can take positive, null, or negative values related respectively to a closed universe, a flat universe and an open universe respectively.

When the gravitational source is a perfect fluid, described by the energy-momentum tensor, the Einstein equations with the metric (2.1) may be given in the second order form

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) \tag{2.2}$$

which represents the dynamic equation and

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G}{3}\rho \tag{2.3}$$

which is a constraint, i.e. it defines the manifold of allowed initial conditions. It takes a simple computation to show that there are no secondary constraints. It constitutes an “invariant relation”, according to Levi-Civita.

From equations (2.2) and (2.3) the first order equation

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + P) \tag{2.4}$$

can be derived by taking the time derivative of (2.3). It can be used alternatively in a system with equation (2.3).

The system of equations (2.2) and (2.3) or (2.3) and (2.4) are not complete, they must be completed by an equation of state $P = P(\rho)$ which is discussed in [20]. Usually a linear equation of state is considered like $P = (\gamma - 1)\rho$ ($\gamma = 1$ is the so-called matter fluid, $\gamma = 4/3$ is the radiation fluid and so on).

Equation (2.4) together with an equation of state, is important for our purpose because it shows that the lefthand side of equations (2.2) and (2.3) can be expressed as a function of a and represents a force in these equations, if we treat them as equations for a “point” particle as a result we have

$$\rho = \frac{\rho_0 a_0^{3\gamma}}{a^{3\gamma}} \tag{2.5}$$

when the equation of state is linear.

It is possible to derive the cosmological model from a point particle Lagrangian, where the expansion factor a takes the part of the particle coordinate. Let us introduce the following Lagrangian[2]

$$\mathcal{L} = 3a\dot{a}^2 - 3ka + 8\pi G\rho_0 a_0^{3\gamma} a^{3(1-\gamma)}. \tag{2.6}$$

The gravitational part is formally derived by substituting directly metric (2.1) into the (field) general relativistic action $\int \sqrt{-g}R$ and the material part is obtained by putting a corresponding potential term $\Phi(a) = 8\pi G\rho_0 a_0^{3\gamma} a^{3(1-\gamma)}$, in the case of a fluid source.

Equation (2.2) follows from the variational method applied to the Lagrangian (2.6).

From equation (2.6) the conjugate momentum of a is

$$p_a = \frac{\partial L}{\partial \dot{a}} = 6a\dot{a}. \tag{2.7}$$

Equation (2.3) is a constraint which is equivalent to the vanishing of the “energy function” $E_{\mathcal{L}}$ associated to the Lagrangian

$$E_{\mathcal{L}} = 3a\dot{a}^2 + 3ka - 8\pi G\rho_0 a_0^{3\gamma} a^{3(1-\gamma)}. \tag{2.8}$$

An alternative way to describe cosmology with a cosmological fluid, with $\Lambda = 0$, was introduced firstly by Lemos [21], Faraoni [23] and also in [2] and [24].

They showed that equations (2.2) and (2.3) can be transformed by means of a reparametrized time in equations similar to the harmonic oscillator ones. By passing to the conformal time η, defined by the relation

$$d\eta = \frac{dt}{a(t)},$$

and with the change of variables

$$w = a^\chi \tag{2.9}$$

where

$$\chi = \frac{3}{2}\gamma - 1$$

equation (2.2) takes the form

$$w'' + k\chi^2 w = 0. \tag{2.10}$$

Similarly it was shown in [3] that cosmological equations with a perfect fluid and a cosmological constant Λ

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) + \frac{2}{3}\Lambda \quad \text{and} \quad \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} \tag{2.11}$$

with the change of variables

$$z = a^{\sigma}, \tag{2.12}$$

where

$$\chi = \frac{3}{2}\,\gamma - 1 \;\text{ and }\;\; \sigma = \,(1+\chi)^{-1} = \frac{2}{3\gamma}$$

are transformed into the equation

$$\ddot{z} = \frac{\Lambda\gamma}{2\sigma}\,z + \frac{k\chi}{\sigma}\frac{1}{z^{2\sigma-1}}. \tag{2.13}$$

or

$$\ddot{z} = \frac{\Lambda}{3}\,z + k\left(1 - \frac{2}{3\gamma}\right)z^{1-(4/3\gamma)}. \tag{2.14}$$

where the time variable is now the cosmic time and not the conformal time as before.

Therefore a flat universe ($k = 0$) with a fluid and a cosmological constant can be regarded again as a harmonic oscillator (anti de Sitter universe), a free particle (Einstein-de Sitter universe) and a repulsive harmonic oscillator (de Sitter universe).

Similar considerations can be done also for cosmological models where the source is originated by a scalar field, which satisfies the Klein-Gordon equation specialized to a homogeneous and isotropic universe. Also in this case, the evolution of universe can be described by equation (2.10). In [27] there are other examples in which the cosmological models with a scalar field can be described by equations similar to (2.10).

3 Evolution in minisuperspace in the framework of tomographic probability representation

We will discuss below the evolution of a universe in the framework of the minisuperspace model discussed in the previous section. Thus the state of the universe is described by a wave function $\Psi(x,t)$. This wave function evolves in time from its initial value $\Psi(x,t_0)$ and this evolution can be described by a propagator $G(x,x',t,t_0)$

$$\Psi(x,t) = \int G(x,x',t,t_0)\Psi(x',t_0)dx'. \tag{3.1}$$

The propagator can be obtained using path integration over classical trajectories of the exponential of the classical action S

$$G(x,x',t,t_0) = \int D[x(t)]e^{\frac{iS[x(t)]}{\hbar}}. \tag{3.2}$$

In our previous work [1] we discussed the properties of the new representation (tomographic probability representation) of the quantum states of the universe.

In this representation (which we discuss below in the framework of a minisuperspace model) the wave function of the universe $\Psi(x,t)$ or the density matrix of the universe

$$\rho(x,x',t) = \Psi(x,t)\Psi^*(x',t) \tag{3.3}$$

can be mapped onto the standard positive distribution $\mathcal{W}(X,\mu,\nu,t)$ of the random variable X depending on the two real extra parameters μ and ν and the time t. The map is given by the formula (we take $\hbar = 1$)

$$\mathcal{W}(X,\mu,\nu,t) = \frac{1}{2\pi|\nu|}\int \rho(y,y',t)e^{i\frac{\mu(y^2-y'^2)}{2\nu}-i\frac{X}{\nu}(y-y')}dy'dy. \tag{3.4}$$

In fact, equation (3.4) is the fractional Fourier transform [28] [25] of the density matrix. The map has inverse and the density matrix can be expressed in terms of the tomographic probability representation as follows

$$\rho(x,x',t) = \frac{1}{2\pi}\int \mathcal{W}(Y,\mu,x-x',t)\, e^{i\left(Y-\frac{\mu}{\nu}(x+x')\right)}dY d\mu. \tag{3.5}$$

The expression (3.4) can be given an affine invariant form [26]

$$\mathcal{W}(X,\mu,\nu,t) = \langle\delta(X-\mu\hat{q}-\nu\hat{p})\rangle \tag{3.6}$$

Here $\langle\ \rangle$ means trace with the density operator $\hat{\rho}(t)$ of the universe state, $\hat{q}$ and $\hat{p}$ are the operators of position (universe expansion factor) and the conjugate moment respectively. From equation (3.6) some properties of the tomogram $\mathcal{W}(X,\mu,\nu,t)$ are easily extracted. First, the universe tomogram is a normalized probability distribution, i.e.

$$\int \mathcal{W}(X,\mu,\nu,t)dX = 1 \tag{3.7}$$

if the universe density operator is normalized (i.e. $Tr\hat{\rho}(t) = 1$). Second, the tomogram of the universe state has the homogeneity property [29]

$$\mathcal{W}(\lambda X,\lambda\mu,\lambda\nu,t) = \frac{1}{|\lambda|}\mathcal{W}(X,\mu,\nu,t) \tag{3.8}$$

The tomogram can be related with such quasidistribution as the Wigner function $W(q,p,t)$ [10] used in the phase space representation of the universe states in [11].

The relation reads

$$\mathcal{W}(X,\mu,\nu,t) = \int W(q,p,t)\delta(X-\mu q-\nu p)\frac{dqdp}{2\pi} \tag{3.9}$$

which is the standard Radon transform of the Wigner function. The physical meaning of the tomogram $\mathcal{W}(X,\mu,\nu,t)$ is the following. One has in the phase space the line

$$X = \mu q + \nu p \tag{3.10}$$

which is given by equating to zero of the delta-function argument in equation (3.9). The real parameters μ and ν can be given in the form

$$\mu = s\cos\theta \qquad\qquad \nu = s^{-1}\sin\theta. \tag{3.11}$$

Here s is a real squeezing parameter and θ is a rotation angle. Then the variable X is identical to the position measured in the new reference frame in the universe phase-space. The new reference frame has new scaled axis sq and $s^{-1}p$ and after the scaling the axis are rotated by an angle θ. Thus the tomogram implies the probability distribution of the random position X measured in the new (scaled and rotated) reference frame in the phase-space. The remarkable property of the tomographic probability distribution is that it is a fair positive probability distribution and it contains a complete information of the universe state contained in the density operator $\hat{\rho}(t)$ which can be expressed in terms of the tomogram as [30]

$$\hat{\rho}(t) = \frac{1}{2\pi}\int \mathcal{W}(X,\mu,\nu,t)e^{i(X-\mu\hat{q}-\nu\hat{p})}dX\,d\mu d\nu \tag{3.12}$$

Formulae (3.6) and (3.12) can be treated with the tomographic star-product quantization schemes [13] used to map the universe quantum observables (operators) onto functions (tomographic symbols) on a manifold (X,μ,ν). The tomographic map can be used not only for the description of the universe state by probability distributions, but also to describe the evolution of the universe (quantum transitions) by means of the standard real positive transition probabilities (alternative to the complex transition probability amplitudes). The transition probability

$$\Pi(X,\mu,\nu,t,X',\mu',\nu',t_0)$$

is the propagator expressed in tomographic representation, it gives the tomogram of the universe $\mathcal{W}(X,\mu,\nu,t)$, if the tomogram at the initial time t_0 is known, in the form

$$\mathcal{W}(X,\mu,\nu,t) = \int \Pi(X,\mu,\nu,t,X',\mu',\nu',t_0)\mathcal{W}(X',\mu',\nu',t_0)dX'd\mu'd\nu'. \tag{3.13}$$

The positive transition probability describing the evolution of the universe has the obvious nonlinear properties used in classical probability theory, namely

$$\begin{aligned}\Pi(X_3,\mu_3,\nu_3,t_3,X_1,\mu_1,\nu_1,t_1) = \int &\Pi(X_3,\mu_3,\nu_3,t_3,X_2,\mu_2,\nu_2,t_2)\\ &\times \Pi(X_2,\mu_2,\nu_2,t_2,X_1,\mu_1,\nu_1,t_1)\,dX_2\,d\mu_2\,d\nu_2.\end{aligned} \tag{3.14}$$

They follow from the associativity property of the evolution maps. This nonlinear relation is the tomographic version of the nonlinear relation of the complex quantum propagators of the universe wave function

$$G(x_3, x_1, t_3, t_1) = \int G(x_3, x_2, t_3, t_2) G(x_2, x_1, t_2, t_1) dx_2. \tag{3.15}$$

Both relations (3.14) and (3.15) imply that the state of the universe evolves from the initial one to the final one through all intermediate states. The remarkable fact is that this quantum evolution of the universe state can be associated with a standard positive transition probabilities like in classical dynamics. This is connected with the existence of the invertible relations of the tomographic and quantum propagators [19][22] . If one denotes

$$K(X, X', Y, Y', t) = G(X, Y, t) G^*(X', Y', t), \tag{3.16}$$

then the quantum propagator may be given the following form

$$K(X, X', Y, Y', t) = \frac{1}{(2\pi)^2} \int \frac{1}{|Y'|} \exp\left\{ i \left(Y - \mu \frac{(X + X')}{2} \right) - i \frac{Z - Z'}{\nu'} Y' \right.$$

$$\left. + i \frac{Z^2 - {Z'}^2}{2\nu'} \mu' \right\} \Pi(Y, \mu, X - X', 0, X', \mu', \nu', t) d\mu \, d\mu' \, dY \, dY' \, d\nu'. \tag{3.17}$$

This relation can be reversed. Thus the propagator for the tomographic probability can be expressed in terms of the Green function $G(x, y, t)$ as follows (we take $t_0 = 0$)

$$\Pi(X, \mu, \nu, X', \mu', \nu', t) = \frac{1}{4\pi} \int k^2 G(a + \frac{k\nu}{2}, y, t) G^*(a - \frac{k\nu}{2}, z, t) \delta(y - z - k\nu')$$

$$\times \exp\left[ik(X' - X + \mu q - \mu' \frac{y + z}{2}) \right] dk dy dz dq. \tag{3.18}$$

The relation can be used to express the tomographic propagation in terms of the Feynmann path integral using the formula for the quantum propagator (3.2) where the classical action is involved. It means that the positive transition probabilities (3.18) can be reexpressed in terms of the double path integral (with four extra usual integrations).

The discussed relations demonstrate that the quantum universe evolution can be described completely using only positive transition probabilities.

Standard complex transition probability amplitudes (and Feynman path integral) can be reconstructed using this transition probability by means of equation (3.17.)

4 Evolution of the universe in the oscillator model framework

As we have shown the equation for the universe evolution in the conformal time picture (2.10) can be cast in the form of an oscillator equation. The oscillator has the frequency $\omega^2 = \pm k\chi^2$.

For $k = 0$ one has the model of free motion. For $k < 0$ one has the model of a inverted oscillator and for $k > 0$ one has the standard oscillator as solution of the equation (2.10). We assume below that the quantum behavior of the universe in the framework of the considered minisuperspace model is described by the quantum behavior of the oscillator as derived in the previous sections. Though the connection (2.9) of the expansion factor $a(\eta)$ with the classical observable w which obeys to oscillator motion provides constraints on the ranging domain of this variable, we assume in the quantum picture of the variable to lie on the real line R. In such approach we apply the tomographic probability representation, developed in the last section, to quantum states of the universe in the framework of the oscillator model. We will denote in the quantum description the variable as q ($w \to q$)and the conformal time as t ($\eta \to t$). Thus the tomographic probability $\mathcal{W}(X, \mu, \nu, t)$of the universe state obeys the evolution equation [1] for the potential energy $V(q)$ in the form

$$\frac{\partial \mathcal{W}(X,\mu,\nu,t)}{\partial t} - \mu \frac{\partial \mathcal{W}(X,\mu,\nu,t)}{\partial \nu} + i \left[V\left(-\left(\frac{\partial}{\partial X}\right)^{-1} \frac{\partial}{\partial \mu} - \frac{i\nu}{2}\frac{\partial}{\partial X} \right) \right.$$
$$\left. -V\left(-\left(\frac{\partial}{\partial X}\right)^{-1} \frac{\partial}{\partial \mu} + \frac{i\nu}{2}\frac{\partial}{\partial X} \right) \right] \mathcal{W}(X,\mu,\nu,t) = 0, \tag{4.1}$$

where the operator $(\partial/\partial X)^{-1}$ is defined by the relation

$$\left(\frac{\partial}{\partial X}\right)^{-1} \int f(y) e^{iyX} dy = \int \frac{f(y)}{(iy)} e^{iyX} dy. \tag{4.2}$$

The propagator of this equation $\Pi(X, \mu, \nu, t, X', \mu', \nu')$ satisfies equation (4.1) with the extra term

$$\frac{\partial \Pi}{\partial t} - \mu \frac{\partial \Pi}{\partial \nu} + i \left[V\left(-\left(\frac{\partial}{\partial X}\right)^{-1} \frac{\partial}{\partial \mu} - \frac{i\nu}{2}\frac{\partial}{\partial X} \right) \right.$$
$$\left. -V\left(-\left(\frac{\partial}{\partial X}\right)^{-1} \frac{\partial}{\partial \mu} + \frac{i\nu}{2}\frac{\partial}{\partial X} \right) \right] \Pi = \delta(\mu-\mu')\delta(\nu-\nu')\delta(X-X')\delta(t), \tag{4.3}$$

For the considered model the general equation for the universe tomogram evolution takes the simple form of a first order differential equation

$$\frac{\partial \mathcal{W}}{\partial t} - \mu \frac{\partial \mathcal{W}}{\partial \nu} + \omega^2 \nu \frac{\partial \mathcal{W}}{\partial \mu} = 0. \tag{4.4}$$

Analogously for the propagator of the tomographic equation for the universe in the framework of the oscillator model one has

$$\frac{\partial \Pi}{\partial t} - \mu \frac{\partial \Pi}{\partial \nu} + \omega^2 \nu \frac{\partial \Pi}{\partial \mu} = \delta(\mu - \mu')\delta(\nu - \nu')\delta(X - X')\delta(t). \tag{4.5}$$

A solution to this equation can be found to be in the case $k > 0$

$$\Pi^{osc.}(X, \mu, \nu, t, X', \mu', \nu') = \delta(X - X')\delta(\mu' - \mu \cos \omega t + \omega\nu \sin \omega t) \times \delta\left(\nu' - \nu \cos \omega t - \frac{\mu}{\omega} \sin \omega t\right). \tag{4.6}$$

In the limit $k = 0$ (free motion) the equation for the tomogram (4.4) becomes

$$\frac{\partial \mathcal{W}(X, \mu, \nu, t)}{\partial t} - \mu \frac{\partial \mathcal{W}(X, \mu, \nu, t)}{\partial \nu} = 0. \tag{4.7}$$

The corresponding propagator solution reads

$$\Pi^{free}(X, \mu, \nu, t, X', \mu', \nu') = \delta(X - X')\delta(\mu' - \mu)\delta(\nu' - \nu - \mu t). \tag{4.8}$$

Finally for the case $k < 0$ the propagator has the form corresponding to a repulsive oscillator

$$\Pi^{rep.}(X, \mu, \nu, t, X', \mu', \nu') = \delta(X - X')\delta(\mu' - \mu \cosh \omega t - \omega\nu \sinh \omega t) \times \delta\left(\nu' - \nu \cosh \omega t - \frac{\mu}{\omega} \sinh \omega t\right). \tag{4.9}$$

Thus we got the dynamics of the universe given by the transition probabilities $\Pi^{osc.}$, Π^{free} and $\Pi^{rep.}$ for the three cases $k > 0$, $k = 0$ and $k < 0$ respectively. One can see that this dynamics is compatible with the dynamics calculated in the standard representation of the complex Green function (quantum propagator). For $k = 1$ the form of the Green function reads

$$G^{osc.}(X, X', t) = \sqrt{\frac{\omega}{2\pi i \sin \omega t}} \exp\left\{\frac{i\omega}{2}\left[\cot \omega t \left(X^2 + {X'}^2\right) - \frac{2XX'}{\sin \omega t}\right]\right\} \tag{4.10}$$

For the case of the free motion model the Green function can be obtained by the limit $\omega \to 0$ in this expression and one has

$$G^{free}(X, X', t) = \sqrt{\frac{1}{2\pi i t}} \exp\left[i\frac{(X - X')^2}{2t}\right] \tag{4.11}$$

and for the repulsive oscillator model one has

$$G^{rep.}(X,X',t)=\sqrt{\frac{\omega}{2\pi i\sinh\omega t}}\exp\left\{\frac{i\omega}{2}\left[\coth\omega t\left(X^2+{X'}^2\right)-\frac{2XX'}{\sinh\omega t}\right]\right\}. \tag{4.12}$$

All these three universe cases can be discussed using the Green function in terms of the Feynmann path integral.

Thus the expression (4.10) is given by the formula

$$G(X,X't)=\int e^{i\int_0^t\left[\frac{\dot{x}^2(t)}{2}-\frac{\omega^2x^2(t)}{2}\right]dt}D[x(t)] \tag{4.13}$$

The integral in the exponent of the path integral provides the classical action for the oscillator

$$S^{cl.}(X,X',t)=\int_0^t\left[\frac{\dot{x}^2(t)}{2}-\frac{\omega^2x^2(t)}{2}\right]dt \tag{4.14}$$

where the trajectories start at $t=0$ at X' and end at time t at the point X. The classical action satisfies the Hamilton-Jacobi equation

$$\frac{\partial S^{cl.}(q,q',t)}{\partial t}+\mathcal{H}\left(q,p=-\frac{\partial S^{cl.}(q,q',t)}{\partial q}\right)=0 \tag{4.15}$$

where $\mathcal{H}$ is the Hamiltonian

$$\mathcal{H}=\frac{p^2}{2}+\frac{\omega^2q^2}{2} \tag{4.16}$$

For the free motion model one has

$$G^{free}(X,X',t)=\int e^{i\int_0^t\frac{\dot{x}^2(t)}{2}dt}D[x(t)]. \tag{4.17}$$

The path integral is integrated and the result (4.11) contains in the exponent term the classical action

$$S^{(f)}(X,X',t)=\frac{(X-X')^2}{2t} \tag{4.18}$$

which is solution of the Hamilton-Jacobi equation with the Hamiltonian

$$\mathcal{H}=\frac{p^2}{2}. \tag{4.19}$$

For the repulsive model one has the same structure of path integral and the result of path integration is expressed in terms of the classical action

$$S^{rep.}(X,X',t)=\frac{\omega}{2}\left[\coth\omega t\left(X^2+{X'}^2\right)-\frac{2XX'}{\sinh\omega t}\right], \tag{4.20}$$

which is solution of the Hamilton-Jacobi equation with the Hamiltonian

$$\mathcal{H}^{rep.} = \frac{p^2}{2} - \frac{\omega^2 q^2}{2}. \tag{4.21}$$

All the obtained propagators complex Green functions or path integrals are related with the propagators in probability representation by means of equations (3.14) and (3.15).

Thus the universe evolution can be described in the oscillator model of minisuperspace for $k > 0$, $k = 0$ and $k < 0$ by means of the standard transition probabilities expressed as propagators $\Pi^{osc.}$, Π^{free} and $\Pi^{rep.}$ respectively.

5 Entropy in cosmological models

With any symplectic tomogram $\mathcal{W}(X, \mu, \nu, t)$ one can associate the tomographic entropy [25] [17]

$$\mathcal{S}(\mu, \nu, t) = -\int \mathcal{W}(X, \mu, \nu, t) \ln \mathcal{W}(X, \mu, \nu, t) dX\,. \tag{5.1}$$

The standard Von Neumann entropy of the quantum state $S_{VN} = -Tr\hat{\rho}\ln\hat{\rho}$ is minimum in the tomographic entropy for a finite Hilbert space. But the Von Neumann entropy does not distinguish any pure state because it is equal to zero for arbitrary $\hat{\rho}_\psi = |\psi><\psi|$. So we hope that the tomographic entropy (5.1) may characterize the chaoticity properties of different states of the universe at least for the cases of minisuperspace models. For example for the oscillator model in the case of a Fock state $|n><n|$ the entropy reads (at a given time moment)

$$\mathcal{S}_n(\mu, \nu) = -\int \left[\frac{1}{2^n}\frac{1}{n!}\frac{\exp(-X^2/(\mu^2+\nu^2))}{\sqrt{\pi}(\mu^2+\nu^2)} H_n^2\left(\frac{X}{\sqrt{(\mu^2+\nu^2)}}\right)\right.$$
$$\left.\times \ln\left(\frac{1}{2^n}\frac{1}{n!}\frac{\exp(-X^2/(\mu^2+\nu^2))}{\sqrt{\pi}(\mu^2+\nu^2)} H_n^2\left(X\right)\right)\right] dX\,. \tag{5.2}$$

For any state of the universe with generic Gaussian Wigner function the tomogram is also a generic Gaussian distribution and it evolves with the time evolution of the universe.

The evolution for the minisuperspace oscillator model is governed by a classical evolution equation. Due to this we hope to compare the entropy values [35] with the numbers obtained from symplectic tomographic entropy. Also we hope to find a relation of the Bekenstein entropy bound with the introduced entropy of the quantum universe state.

The properties of the tomographic entropy are studied in [36][37].

Since the tomogram of a quantum state satisfies the homogeneity condition (3.8) the tomographic entropy has the property [37]

$$S(\sqrt{\mu^2+\nu^2}\cos\theta, \sqrt{\mu^2+\nu^2}\sin\theta, t) - \frac{1}{2}\sqrt{\mu^2+\nu^2} = f(\theta, t) \tag{5.3}$$

We used the polar coordinates

$$\begin{aligned} \mu &= \sqrt{\mu^2+\nu^2}\cos\theta \\ \nu &= \sqrt{\mu^2+\nu^2}\sin\theta. \end{aligned} \tag{5.4}$$

Also, the particular values of the entropy

$$S(1,0,t) = -\int \rho(x,x,t) ln[\rho(x,x,t)]dx \tag{5.5}$$

and

$$S(0,1,t) = -\int \rho(p,p,t) ln[\rho(p,p,t)]dp \tag{5.6}$$

are the Shannon entropies [31][32] connected with the probability distribution densities in position and momentum respectively.

It is known [33] [34] that the entropies satisfy the following inequality

$$S(1,0,t) + S(0,1,t) \geq \ln(\pi e). \tag{5.7}$$

This inequality was extended [36][37] to give the inequality for the tomogram of the quantum state

$$\int \left[\mathcal{W}(X,\theta,t)\ln\mathcal{W}(X,\theta,t) + \mathcal{W}(X,\theta+\frac{\pi}{2},t)\ln\mathcal{W}(X,\theta+\frac{\pi}{2},t)\right]dX + \ln(\pi e) \leq 0. \tag{5.8}$$

Here

$$\mathcal{W}(X,\theta,t) = \mathcal{W}(X,\mu=\cos\theta,\nu=\sin\theta,t). \tag{5.9}$$

The inequality (5.8) can be used to check whether the tomogram $\mathcal{W}(X,\mu,\nu,t)$ satisfies the quantum constraints or not. One can check that the entropy of an excited state of the universe in the framework of the oscillator minisuperspace model given by eq. (5.2) satisfies the above inequality. In this case since the angle θ disappears from the lefthand side of eq.(5.2) the inequality takes the form of an integral inequality for the Hermite polynomial.

$$\frac{1}{2^n\sqrt{\pi}}\frac{1}{n!}\int\left[e^{-X^2}H_n^2(x)\ln\frac{1}{2^n}\frac{1}{n!\sqrt{\pi}}e^{-X^2}H_n^2(X)\right]dX + \frac{1}{2}\ln(\pi e) \leq 0. \tag{5.10}$$

In the case of the Gaussian coherent states the inequality is saturated and becomes equality. For example, the ground state with wave function

$$\psi_0 = \frac{e^{-\frac{q^2}{2}}}{\sqrt[4]{\pi}} \tag{5.11}$$

the tomogram reads $\mathcal{W}(X, \cos\theta, \sin\theta) = e^{-X^2}/\sqrt{\pi}$ and

$$\frac{1}{\sqrt{\pi}} \int e^{-X^2} \ln \frac{e^{-X^2}}{\sqrt{\pi}} + \frac{1}{2} \ln(\pi e) = 0. \tag{5.12}$$

The discussed inequalities called entropic uncertainty principle are connected with the Heisenberg uncertainty relations. For Gaussian coherent states (or ground state) the entropic inequality (5.8)is equivalent to

$$(\delta q)^2 (\delta p)^2 \geq \frac{1}{4} \tag{5.13}$$

where $(\delta q)^2$ and $(\delta q)^2(\delta p)^2$ are the dispersion of the position and the momentum when are both equal to $1/2$.

Calculating the entropies $S(1,0)$ and $S(0,1)$ for Gaussians with these dispersions one can find that the entropic inequalities in this case provide the inequality (5.13). Thus the quantum Heisenberg uncertainty relation can be cast for the universe in the framework of minisuperspace model as constraint for the universe state tomogram. Thus, if one can extract some experimental data on the universe state tomogram the tomographic entropy can be used to control the compatibility of the quantum gravity constraints with the observable data. It is worthy to note that only the quantum tomogram must satisfy the discussed inequality. On the other side the classical state may not to satisfy it. The inequality (5.8) can be written in another form. In fact, in the lefthand side of this inequality we have the periodic function of the angle θ. Expanding this function into Fourier series we get

$$\int \left[\mathcal{W}(X,\theta,t) \ln \mathcal{W}(X,\theta,t) + \mathcal{W}(X,\theta + \frac{\pi}{2}, t) \ln \mathcal{W}(X, \theta + \frac{\pi}{2}, t) \right] dX = \sum c_m(t) e^{im\theta}. \tag{5.14}$$

Averaging the function over the angles θ we obtain the inequality

$$c_0(t) + ln(\pi e) \leq 0. \tag{5.15}$$

Here the constant contribution to the Fourier series depending on time is the quantum state characteristic which is the functional of the quantum tomogram of the universe. We discussed the one mode case.

The number $\ln(\pi e) = 2.14$ can be interpreted as the "entropy" of the vacuum state of the mode. In fact, if one takes the ground state of the harmonic oscillator, the Shannon entropy associated to this state using the probability distribution in position equals the Shannon entropy associated to the distribution in momentum. Each of these equal entropies read $S_x = S_p = 1/2 + \ln(\pi)$.

So we can associate the positive minimal entropy $\ln(\pi e)$ to the vacuum state analogously to the minimal ground state energy. This energy creates the notion of Casimir energy for many modes.

We can suggest that this is the analog of the vacuum dimensionless entropy associated to the $\ln(\pi e)$ term for each mode.

Some important inequalities for entropy related to gravity have been discovered by Bekenstein [38] and by Verlinde [39], The constant c_0 is the mean value of the sum of Shannon entropies of the probability distributions for two conjugate variables (position and momentum). In classical mechanics there is no correlations between these two entropies. In quantum approach there appeared such kind of correlation expressed in terms of the inequality (5.15). For the case of several degrees of freedom the bound of the inequality takes an integer factor. becoming $N \ln \pi e$ where N is the number of degrees of freedom. Thus the entropy bound is "quantized" depending on the number of degrees of freedom corresponding to the minisuperspace model.

A straightforward calculation shows that we can also produce the entropy evolution by substituting equation (3.13) into equation (5.1). We find

$$\mathcal{S}(\mu,\nu,t) = -\int\int \Pi(X,\mu,\nu,t,X',\mu',\nu',t_0)\mathcal{W}(X',\mu',\nu',t_0)dX'd\mu'd\nu'$$

$$\times \ln \int \Pi(X,\mu,\nu,t,X',\mu',\nu',t_0)\mathcal{W}(X',\mu',\nu',t_0)dX'd\mu'd\nu'dX\,. \qquad (5.16)$$

which can be compared with the initial entropy

$$\mathcal{S}(\mu,\nu,t_0) = -\int \mathcal{W}(X,\mu,\nu,t_0)\ln \mathcal{W}(X,\mu,\nu,t_0)dX\,. \qquad (5.17)$$

Thus the tomographic transition probability (propagator) determines the evolution of the tomographic entropy. The form of the propagator is compatible with the constraints (inequalities) which must be fulfilled for the tomographic entropy and for the tomogram.

6 Conclusions

To conclude we discuss the main results of the work. In addition to what suggested in [1], the probability representation of the universe quantum states for which the states (e.g. of the universe in a minisuperspace model) are described by the standard positive probability distribution, we introduce the description of the universe dynamics by means of standard transition probabilities.

The transition probabilities are determined as propagators (integral kernels) providing the evolution of the universe tomograms. It is shown that there is a relation of the standard propagator determining the quantum evolution of the universe wave function to the tomographic propagator. This relation permits to reconstruct the complex propagator for the wave function in terms of the positive propagator for the universe tomogram. Also, one can express the propagator for the tomogram in terms of the propagator for the wave function of the universe.

These relations between the propagators mean that the Feynmann path integral formulation or the universe properties (in quantum gravity) contains the same information that the probability representation of the quantum states of the universe including the universe quantum evolution. As the simplest example of the suggested transition probability picture, we considered the minisuperspace model for which classical and quantum evolution is described by the harmonic vibrations in conformal time [21], [23], [27], [24]. The specific property of this minisuperspace model is that the tomographic propagators for both classical and quantum universe tomograms coincide. This fact provides some possibility to connect observations related to today classical epoch of the universe and its purely initial quantum state. In the framework of the suggested approach (and in the framework of the considered oscillator model), the universe evolution can be studied using specific properties of the tomographic propagator. If one considers tomograms and their evolution in classical mechanics [17][19] the specific property of the linear systems (e.g. oscillator model) is that the tomographic propagators in quantum and classical domains are in one-to-one correspondence and are given in the same carrier space, therefore we may say that in this picture the difference of the quantum and classical evolution is in the initial conditions, and relies on the fact that the product of functionals on the "wave functions" are multiplied pointwise in the classical picture and non-locally in the quantum picture. This non-local product contains all the information of the indetermination relations and consequent constraints on the allowed tomograms.

They must satisfy uncertainty relations. The choice of initial conditions (initial tomogram of the universe) in correspondence with the uncertainty relation provides a possibility to avoid the singularity of the metric, which is unavoidable in the classical picture. But the following evolution of the universe coded by the tomographic propagator is the same (for the oscillator model).

Due to this result, one can extract from the present observational classical data conclusions over the cosmological evolution. Evolving backwards in time the present situation by means of the "true" quantum or the classical propagators, we may find discrepancies between the initial conditions at minus infinity. Using the notion of tomographic entropy we also established the specific constraints for the quantum states of the universe expressed in terms of an inequality. This inequality must be satisfied in the quantum mechanical approach and it can be violated in the classical approach.

This inequality provide a lower bound for the quantum entropy of the universe.

We think that there can be a relation of this lower quantum bound with the lower bounds for entropy discussed in [39].

We are going to discuss this aspect in a future work.

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